UDC 371.3::51-057.874
159.955.5:51
Иновације у настави, XXVIII, 2015/1, стр. 42–50

Received: 11. 10. 2014.
Accepted: 20. 1. 2015.

Survey Paper

Radovan Antonijević[1], PhD,
Nataša Vujisić Živković, PhD
University of Belgrade, Faculty of Philosophy

Influence of interconnecting knowledge in mathematics teaching on development of mathemathical thinking

Summary*. This paper is on the basic features of integration of knowledge in mathematics teaching and contribution of this process to development of mathematical thinking of students. The standpoint presented here is that it is of great importance for students in mathematics able, with help of appropriate teaching content, to adopt a system of interconnected and conditioned mathematical knowledge and concepts. The knowledge system students can form in the teaching of mathematics, under certain conditions, if the selection of teaching contents allow its formation in the learning process represents more stable and logically consistent system of knowledge in comparison to any other area of learning in the classroom. The basis of the connectivity of knowledge in mathematics, according to the basic assumptions of the theory of developmental teaching, is the discovery of the subject of starting basic mathematical concepts in mathematics teaching, such as the relations among size, number, number system, set and others. Practicing different cognitive activities of acquiring mathematical concepts and interconnecting knowledge in mathematics teaching contribute to the development of mathematical thinking in students.*

Key words*: mathematics teaching, interconnectedness of knowledge, activities of interconnecting knowledge, mathematical thinking.*

Mathematics as a science is one of the most developed and organized fields of science, while mathematics in today's conditions is further developing intensively. All of these are followed by a significant improvement of the scientific methodology of studying in this area. Mathematical knowledge and concepts, as a product of the process of mathematics education, become irreplaceable and indispensable part of modern institutionalized education of every individual, regardless of the level and type of education that an individual acquires. Direct or indirect applying of mathematical knowledge is spreading and deepening, and there is no area of human life and work where the use of appropriate mathematical knowledge, abilities and skills are not required, to a greater or lesser extent. Yet another significant fact is that, in addition to the applying of mathematical knowledge and mathematical op-

1 aa_radovan@yahoo.com

erations, applying of the methodology of mathematics and mathematical thinking and different forms of learning in mathematics, as well as various mathematical procedures are typical for mathematics as a science.

Connections and relationships that exist between different mathematical skills and concepts, as well as between different areas of mathematics, are by their logical basis something more significant, deeper and more stable in comparison to the connections of concepts, laws, rules and other forms of knowledge that exist within other scientific fields and disciplines. Forms of knowledge in mathematical sciences are an integral part of a complete and logically consistent system, stable structure, which is more stable and logically consistent, in relation to system of knowledge in any other field of science. Based on the above reasons, it can be concluded that the system of mathematical concepts, laws, rules, axioms, theorems, mathematical operations, mathematical formulas, equations, mathematical procedures and other relevant forms of mathematical knowledge, is undoubtedly *a role model*, an ideal system of knowledge in science. Such a paradigmatic model of knowledge system is more complex in comparison to any other scientific discipline, by the process, organization and reorganization of the fund of knowledge and different theoretical systems and sub-systems of knowledge. Mathematical science and its system, in this sense, is the real core, the foundation of all processes related to the formation of a specific system of knowledge that takes place in particular scientific fields and disciplines.

System of mathematical knowledge in teaching

Starting from the importance of mathematics education of every individual, within didactics and methodology of mathematics, it is increasingly pointed at the necessity of setting up a stable foundation in mathematics education, even in lower grades of elementary school, which would serve as a basis for successful mastery of mathematical content in higher grades of elementary school, in secondary school and beyond. In addition, the necessity of students to master *mathematical approach to objective reality* and the basic elements of mathematical thinking in lower grades is emphasized. Cvetković (1981: 69) writes about this: "The teaching of mathematics in primary school, especially in lower grades, lays the foundations of more complete mathematics education. It has to introduce students to specific, mathematical approach to objects, phenomena and processes that enable them to successfully *discover subject of mathematics*. This approach is new and unusual for students. In teaching, however, they must master it" (emphasis by R.A.). Focus of mathematics teaching is to discover the subject of mathematics, its underlying basis, and it is the major objective of mathematics teaching, whose implementation must be subordinated to the content of mathematics teaching. Realization of this objective is of the crucial importance for the creation, expansion and deepening of the system of mathematical knowledge, development of a mathematical approach to objective reality, and development of a kind of mathematical view of objective reality.

Along with the establishment of the system of mathematical knowledge and concepts, *mathematical thinking is developed* in students in the learning process, or through the contents of mathematics (Antonijević, 2000). Under certain conditions related to the nature of mathematical knowledge and concepts within the content of the course, it is possible to achieve a two-way influence in the process of learning. This implies necessity of connection between learning and development of thinking operations in teaching within the process of learning in mathematics. When these two developmental courses are allowed to be relatively independent within the learning process, it is possible in practice that mathematical knowledge and mathematical thinking become two dimensions of a unique process of discovering the subject of mathematics.

Based on these opinions, it can be concluded that one of the key objectives of teaching mathematics should be a just establishment of the system of mathematical knowledge and concepts in students. It is believed that this knowledge system appears to be a stable basis for the attainment of knowledge in the areas that are directly related to mathematics, based on the development of the entire system of knowledge and concepts that would enable better understanding and adoption of new scientific knowledge by students in the learning process. Development of such a system of knowledge would be accomplished as part of a single process, and through the course of development of mathematical thinking of students in classroom, although an attitude on mutual influence and interpenetration of knowledge and thinking is not clearly defined, nor has it explained some of the mechanisms of this kind of influence in mathematics teaching.

Good and Brophy wrote that the American National Council of Mathematics Teachers had issued the instruction for the teaching of mathematics in 1988, which stressed the importance of teaching students to understand and apply high level mathematical knowledge. In this instruction, the main emphasis is placed on developing students' *mathematical power*, which refers to their ability to explore, intuitively conclude, think logically, and to effectively use different mathematical models for solving specific, non-routine problems (Good & Brophy, 1991: 456). The term "development of mathematical power" is based on the fact that mathematics is something more than ordered set of concepts and skills that students need to be taught. According to these authors, mathematics includes research methods and reasoning, communication tools and concepts of logical connections. Based on these reasons, the development of the mathematical power of each individual needs to include development of mathematical thinking's abilities.

Traditional approach of development of mathematical concepts in teaching

Content analysis of textbooks of mathematics in the lower grades, as well as teaching methods in mathematics, shows obvious presence of the traditional empiricist frame of concept acquisition and development of thinking operations in mathematics. This is easily demonstrated by the example of adoption of the concept of "number", one of the fundamental concepts in mathematics, which is crucial to a student's deeper and more comprehensive understanding of the subject of mathematics as a science. The adoption and development of this concept in mathematics teaching in lower grades is important for successful mastering of the other basic mathematical concepts.

In mathematics teaching based on traditional empiricist conception of knowledge attainment, process of adoption of the concept of "number" is supported by activities of the observation of different types of physically separated objects, as well as the observation of a group of physical objects, in certain ways. In such an organized cognitive process, the emphasis is placed on the activities of direct observation and perception of external sensory-obvious properties of these objects. In this kind of activities students are directed to identify their separateness, one from another. Thus, the mathematics textbook for first grade shows images of different objects or animals, and students are asked to nominate how many of these objects or animals they see through the illustrations in the book. And with the continued use of various auxiliary objects such as cubes, spheres, sticks, etc., it is practically proved to students that, for example, the number "1" means the same as "one cube", "one ball" or "one stick". Writing about the basic characteristics of the process of adopting the concept of number in mathematics teaching based on empiricist conception of concepts, Davydov (1972: 152) points out that this scheme introduces a number to a student by following external characteristics: "By comparing quali-

ty of many different objects, a student finds something general in them while indicating separateness of cases, how the objects differ from one another, including some of their space and time limitations. This is an individual case and each case contains such a singularity, distinctiveness, which is external and can be perceived". The concept of "number one" is formed when this external sensory-evident property is accepted besides a group of other properties that individual items have. All other external characteristics of the observed objects are ignored and discarded, while a characteristic physical separateness of the observed individual cases is retained. Cvetković (1981: 73) points out that in the process of adopting the concept of "number 1" as the number "1" is "fixed and abstracted individuality as common external property of different individual objects or sets that contain any, but always a single object. In this way, the common external properties of the observed objects are taken as the important properties, and are kept within the content of the formed concepts.

In the initial teaching of mathematics, as it is presented in theoretical conceptions of concept attainment in teaching, also presented in the methodology of teaching mathematics, there is the presence of traditional empiricist orientation of concept attainment and development of thinking in the classroom. Cvetković (1981: 74) emphasizes that teaching methodology as a result of epistemological-logical theory on which it is based, is not explored and does not reveal the origin and formation of the content of concept of "number", and its internal underlying basis. For these reasons, students in the process of learning are not provided with the formation of appropriate actions that will enable them to discover the object of the concept of "number" itself. Retention of students at this level of introduction to the properties of numbers in initial teaching of mathematics doesn't enable students to adequately start accessing a proper start of accession the world of mathematics, by learning about the essence of mathematical objects.

In current practice of the realization of mathematics certain shortcomings in the curriculum, textbooks and process of teaching mathematics itself can be identified in teaching. These shortcomings are related to the nature of the content of mathematics teaching in the lower grades, their role and the possibility of using them to attain the key mathematical concepts. Prvanović (1970: 36) sees these failures, that are manifested in the cognitive processes in mathematics teaching, as failures with major consequences for the development of the mathematical knowledge and the development of mathematical thinking in students. He characterizes these failures in a way that it is easier to take adolescent mathematically if he has never had contact with math, than if they learned mathematics in traditionally conceived programmes, so in this case a large number of children remain eternally "invalid" for mathematics, and only in certain cases the exiled ability is returned. He also believes that "disability" for mathematics is a consequence of the nature of the content of teaching in traditionally oriented teaching and that the only "cure" in this situation is its radical reform, and reform of teaching contents.

The so-called cognitivistic and constructivist orientation within developmental psychology, co-founded by Piaget, says that the problems which are the subject of discussion and study within this theoretical orientation are logical and mathematical structures in individual systems, operations specific to these structures, ways of their formation and development, as well as the possibility to influence their emergence and development at certain developmental ages. One of the most important debates on polygons of various scientific opinions and arguments in this area relates to the problem of the nature of these structures themselves. In this regard, according to the accepted starting point, certain understanding of the possibilities for the systematic influence of teaching on the development of logical and mathematical structures in students is developed and they are related to the age of students

at which the influence on the development of these structures is possible.

Piaget (1983) points out that one of the important conclusions that he took out of his studies in psychology of intelligence is that logical-mathematical structures are not something children innately know, but bit by bit, these structures are being constructed through development. At Piaget, logical-mathematical structures are functional systems of multiple interconnected and mutually conditioned logical-mathematical operations, which is one of the fundamental properties of reversibility, the ability to run in both directions.

Mathematics teaching as an integral process of development of mathematical concepts and mathematical thinking

Davydov points out (1986; 1996) that the system of numbers in students is based on *structure of mathematical concepts*, which is the basis of a system of their mathematical concepts and operations. The concept "number" is associated with many concepts that precede it, but it is particularly associated with the "set", "equivalence" and "scope" concepts. In this way, according to Davydov, the number in general construction of modern mathematical concepts does not appear as the starting and basic concept, although there are certain views (Shevchenko), according to which the concept of number is root mathematical concept in the true sense of the word, based on the premise that it can not be directly defined using other terms. Davydov believes that some important concepts such as "set", "size", "scope" and "group", leading to a number and independently of it, and the properties of the systems of numbers themselves may be disclosed on the basis of other common mathematics concepts.

Within years of research under guidance of Davydov and Elkonin (1966), the basis of the experimental mathematics teaching is made by *concept of a real number*. Unlike mathematics curriculum in traditional education, it has provided such introductory section, in which students especially studied genetic basis of subsequent generating of all aspects of the real number, especially when studying the concept of "size". Such an approach to the contents of experimental programme of the subject of mathematics determines the following system of its basic teaching objectives, especially designed for the use in lower grades: (1) introducing students to the sphere of relations between size and formation of abstract mathematical concept of size; (2) students' discovery of the relations of divisibility of size as general property of number and forming an abstract concept of number with them and understanding of the basic connections among its components; (3) gradual introduction of students in the area of special different types of numbers (natural, negative numbers and fractions) and forming the concepts of these numbers in them; (4) discovering a kind of univalent structure of mathematical operations by students and forming, within them, understanding of the interaction of elements of the basic arithmetic operations (Davydov, 1986: 180).

One of the implicit mathematical objectives, according to Cvetković (1981), is to enable students to see the world from a *mathematical point of view*, and at the same time, this objective of mathematics teaching is also an important condition for their basic, general orientation to the mathematical reality, which represents itself an element of connecting mathematical knowledge and concepts, considering the fact that it is a general orientation to the mathematical reality. According to Cvetković (1981), adopting a proper mathematical concepts which by nature of their interconnections and relationships allow students to create logically consistent system of knowledge in this area, has a key role in formation of the system of mathematical knowledge. On the other hand, there is the emphasis on the relationship of dependency between development of mathematical concepts and development of mathematical thinking, a requirement which would

be necessary to establish through the content of the mathematics teaching.

Analyzing a kind of discrepancy between social and professionally established educational objectives and development of mathematical sciences, on the one hand, and teaching methodology solutions of teaching process and outcomes of educational process in the form of success that is achieved in classroom, on the other hand, Cvetković (1981: 70) emphasizes that there is a marked delay, and sometimes contradiction between contents of certain concepts in teaching and the content of these concepts in science, and that there is an unsatisfactory achievement in domain of understanding, adoption and applying of mathematical concepts, inadequate teaching influence on development of intellectual abilities of students, which is particularly reflected on the level and potential for development of mathematical thinking. This indisputable fact specifically refers to the mathematical concepts that are adopted in mathematics teaching in initial grades of primary school, which is based on adopting empiricist knowledge and concepts in this field. Recognizing the fundamental process of teaching based on traditional empirist conception of concept and development of thinking in classroom, and the need to overcome existing shortcomings in the teaching of mathematics, Cvetković (1981: 71) points out the following: "It is necessary to change the contents and methods of teaching in order to create conditions for *development of theoretical scientific thinking in students* from beginning of schooling. Only then is it possible to establish the necessary balance between modern state and trends of science and scientific thinking and success that in this respect is expected in the classroom". Furthermore, he believes that the realization of these objectives is a very difficult and complex objective, which assumes specialized and systematic theoretical and experimental research. The purpose of this research would consist of its contribution to constitution of curricula in mathematics, as a kind of new basis, which would include selection of contents of mathematics teaching, which would allow systematic influence on the formation of mathematical knowledge system and the development of mathematical thinking in students.

Considering the current situation in the teaching of mathematics, when it comes to the use of different teaching methods, Prvanović points out that the existing mathematics teaching mainly represents *exhibiting*, *pointing* and *dialogic methods*. Despite the fact that a primary school does not build axiomatic mathematical structure, according to Prvanović (1970: 62), mathematics education of students can be realized "by the combination of various methods, which allows each student to independently *build mathematical structures*, and therefore mental structures", and he adds that this can be accomplished by using the *method of problems and methods of problem situations*, in the following manner: (1) "placing" students in situations that allow them to own speculative activity, observe and select what is mathematical; (2) guiding students through math by different situations, so that they "see" that "many things can be called by the same name"; and (3) progressive training to structure situations (1970: 62-63). In this case, the problem-solving tasks in mathematics, in which dialogic methods and methods of problem situations are highlighted, are seen as means of active learning of concepts in mathematics and, also according to Prvanović, means of influencing the construction of mathematical and mental structures in students. It is pointed out that the basis of connecting knowledge is, above all, independent thinking activity of students focused on specific mathematical content.

According to Prvanović, bases of development of a system of mathematical concepts in students is just development of thinking type in mathematics teaching, which he referred to as *mathematical thinking*. Explaining the importance of the development and function of mathematical thinking among students, he points out the following: "In the shortest and most concise way, we can say that

mathematical thinking constructs mathematical concepts, operates mathematical concepts and discovers mathematical relationships and dependencies among mathematical (and non-mathematical) concepts, that is, it reveals the mathematical truths (facts)" (1970: 14). According to him, one of the most important objectives of mathematics teaching is to enable students to think mathematically and to "know" that it is the properties of contemporary mathematical thinking, that allow them entering *the contemporary mathematics* (1970: 14).The realization of this objective in the mathematics teaching makes it possible to connect mathematical knowledge and concepts in students, in terms of creating a system of logically related knowledge and concepts in the field of mathematics. This author also emphasizes the need that the mathematics teaching should enable mastery of the subject of modern mathematics. In teaching of mathematics, this can be enabled by achieving systematic influence in order to precipitate development of the bases of mathematical thinking in students.

Conclusion

Knowledge system which may be formed in students in mathematics teaching, if the selection of teaching contents allow its formation in the learning process, is more stable and logically consistent system of knowledge in comparison to any other area of learning in classroom. The essential basis of internal homogeneity of this type of knowledge and concepts in students is made by the nature and basic properties of the interconnectedness and interdependence of knowledge and concepts within mathematical sciences. In this regard, it is necessary to overcome the key shortcomings of the modern mathematics teaching, especially deficiencies related to the nature of teaching contents and the nature of knowledge and concepts that are selected. It is necessary, through the selection of appropriate contents of mathematics teaching, to enable students through the process of learning in classroom to reveal the nature of the content of mathematics as a science, adopting the right scientific and mathematical concepts, and at the same time developing mathematical thinking and mathematical view of the world.

References

- Antonijević, R. (2000): Naučno-teorijski pojmovi kao osnova sadržaja nastave [Scientific theoretical concepts as base of teaching contents], *Pedagogija*, br. 3-4, 455-460.
- Cvetković, Ž. (1981): Neka novija shvatanja o usvajanju matematičkih pojmova u osnovnoj školi [Some new views on attainment of mathematical concepts in elementary school], *Nastava i vaspitanje*, br. 1, 69-79.
- Cvetković, Ž. (1982): *Usvajanje pojmova u nastavi* [Attainment of concepts in teaching]. Beograd: Zavod za udžbenike i nastavna sredstva.
- Cvetković, Ž. (1995): Uloga opštih znanja u saznavanju posebnog i pojedinačnog [Role of common knowledge in cognizing particular and specific elements of knowledge], *Saznavanje i nastava* (171-194). Beograd: Institut za pedagoška istraživanja.
- Давыдов, В.В. (1966): Психологические особенности »дочисловового« периода обучения математике [Psychological characteristics of pre-number period of mathematics teaching]; в Д.Б. Эльконин и В.В. Давыдов (ред.): *Возрастные возможности усвоения знаний* (104-189). Москва: »Просвещение«.
- Давыдов, В.В. (1972): *Виды обобщения в обучении: Логико-психологические проблемы построения учебных предметов* [Forms of generalizations in teaching: problems of constituting of teaching subjects]. Москва: »Педагогика«.

- Давыдов, В.В. (1986): *Проблемы развивающего обучения: опыт теоретического и экспериментального психологического исследования* [Problems of developmental teaching: conception of theoretical and experimental psychological study]. Москва: »Педагогика«.
- Давыдов, В.В. (1996): *Теория развивающего обучения* [Theory of developmental teaching]. Москва: Российская Академия образования.
- Эльконин, Д.Б. и В.В. Давыдов (ред.) (1966): *Возрастные возможности усвоения знаний* [Age opportunities of knowledge attainment]. Москва: »Просвещение«.
- Good, T.L. & J.E. Brophy (1991): *Looking in classrooms*. New York: Harper Collins.
- Milanović-Nahod, S. (1988): *Kognitivne teorije i nastava* [Cognitive theories and teaching]. Beograd: Prosveta.
- Pijaže, Ž. (1983): *Poreklo saznanja – Studije iz genetičke epistemologije* [Origins of knowledge – Studies in genetic psychology]. Beograd: Nolit.
- Prvanović, S. (1970): *Metodika savremenog matematičkog obrazovanja*. Beograd: Zavod za izdavanje udžbenika Socijalističke republike Srbije.
- Prvanović, S. (1971): *Moderna matematika* [Modern mathematics]. Beograd: Zavod za izdavanje udžbenika Socijalističke republike Srbije.

др Радован Антонијевић, др Наташа Вујисић Живковић
Универзитет у Београду, Филозофски факултет

Утицај повезаности знања у настави математике на развој математичког мишљења

Резиме: *У овом раду разматрају се основне карактеристике повезаности знања у настави математике и њихова повезаност са развојем математичког мишљења код ученика. Везе и односи који постоје између различитих математичких знања и појмова, као и између различитих области математике, представљају по својој логичкој основи нешто значајно више, дубље и постојаније у односу на повезаност појмова, закона, правила и других облика знања, који постоје у оквиру других научних области и научних дисциплина. Облици знања у математичкој науци саставни су део целовитог и логички доследног система (постојане структуре) који је стабилнији и логички доследнији у односу на систем знања у било којој другој области науке. На основу наведених разлога може се закључити да систем математичких појмова – закона, правила, аксиома, теорема, математичких операција, математичких формула, једначина, математичких поступака и других релевантних облика математичких знања – представља несумњиво узорни модел, односно идеал система знања у науци. Таквом једном парадигматском моделу система знања теже све посебне научне дисциплине, у процесу конституисања, организовања и реорганизовања фонда знања и различитих теоријских система и подсистема знања. У вези са формирањем специфичног*

система знања, математичка наука и њен систем, у том смислу, представљају прави ослонац, темељ свих процеса који се одвијају у појединим научним областима и дисциплинама.

Заступљено је становиште да је од изузетног значаја да ученицима у настави математике, помоћу адекватних садржаја наставе, буде омогућено да усвајају систем међусобно повезаних и условљених математичких знања и појмова. Систем знања који се под одређеним условима може код ученика формирати у настави математике, уколико се избором садржаја наставе омогући његово формирање у наставном процесу, представља стабилнији и логички доследнији систем знања у односу на било коју другу област сазнавања у настави. Основу процеса повезивања знања у настави математике, према основним поставкама теорије развијајуће наставе, представља откривање у настави математике саме предметне основне исходишних математичких појмова, као што су однос између величина, број, систем бројева, скуп и други. Суштинску основу унутрашње хомогености ове врсте знања и појмова у сазнању ученика треба да чини природа и основна својства међусобне повезаности и условљености знања и појмова у оквиру математичке науке. У том смислу, неопходно је превазићи кључне недостатке савремене наставе математике, поготову недостатке који се односе на природу садржаја наставе и природу знања и појмова који су изабрани. Неопходно је путем избора адекватних садржаја наставе омогућити ученицима да у процесу сазнавања у настави открију саму природу предметног садржаја математике као науке, усвајајући праве научне математичке појмове и у исто време развијајући математичко мишљење и математички поглед на свет.

Упоредо са формирањем система математичких знања и појмова развија се и математичко мишљење код ученика у наставном процесу, односно путем садржаја наставе математике, под одређеним условима који се односе на природу математичких знања и појмова у оквиру садржаја наставе, могуће је остварити двосмерни утицај у процесу сазнавања. То подразумева да се у процесу сазнавања у настави математике повежу усвајање знања и развој мисаоних операција. Када се у наставном процесу омогући да та два развојна тока иду упоредо и у односу међузависности, тада је практично омогућено да математичко знање и математичко мишљење представљају две димензије јединственог процеса откривања предмета математике.

На основу наведених ставова може се закључити да један од кључних задатака наставе математике треба да буде формирање система математичких знања и појмова код ученика. Сматра се да би овај систем знања чинио стабилну основу усвајања знања у областима које су са математиком непосредно повезане, основу развоја укупног система знања и појмова, која би ученицима омогућила боље разумевање и усвајање нових научних знања у наставном процесу. Упоредо са развијањем једног оваквог система знања остваривао би се, као саставни део јединственог процеса, и ток развоја математичког мишљења ученика у настави.

Кључне речи: *настава математике, повезаност знања, активности повезивања знања, научно-теоријска математичка знања.*